

HERMITIAN GEOMETRY ON RESOLVENT SET

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ABSTRACT. For a tuple $A = (A_1, A_2, \dots, A_n)$ of elements in a unital Banach algebra \mathcal{B} , its *projective joint spectrum* $P(A)$ is the collection of $z \in \mathbb{C}^n$ such that $A(z) = z_1 A_1 + z_2 A_2 + \dots + z_n A_n$ is not invertible. It is known that the \mathcal{B} -valued 1-form $\omega_A(z) = A^{-1}(z)dA(z)$ contains much topological information about the joint resolvent set $P^c(A)$. This paper defines Hermitian metric on $P^c(A)$ through the \mathcal{B} -valued *fundamental form* $\Omega_A = -\omega_A^* \wedge \omega_A$ and its coupling with *faithful* states ϕ on \mathcal{B} . The connection between the tuple A and the metrics is the main subject of this paper. A notable feature of this metric is that it has singularities at the joint spectrum $P(A)$. So completeness of the metric is an important issue. When this construction is applied to a single operator V , the metric on the resolvent set $\rho(V)$ adds new ingredients to functional calculus. An interesting example is when V is quasi-nilpotent, in which case the metric live on the punctured complex plane. It turns out that the blow up rate of the metric at the origin 0 is directly linked with V 's lattice of hyper-invariant subspaces.

0. INTRODUCTION

In [6], the first author and Cowen introduced geometric concepts such as holomorphic bundle and curvature into Operator Theory. This gave rise to complete and computable invariants for the Cowen-Douglas operators. This idea was followed up in a series of papers in the study of Hilbert modules in analytic function spaces, where a curvature invariant is defined for some natural tuples of commuting operators. We refer readers to [12, 13, 14] and the references therein for this line of

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research. This paper aims to study general (possibly non-commuting) tuples using geometric ideas. Its approach is to use the newly emerged concept of projective joint spectrum $P(A)$ introduced by the second author (cf. [39]).

Let \mathcal{B} be a complex Banach algebra with unit I and $A = (A_1, A_2, \dots, A_n)$ be a tuple of linearly independent elements in \mathcal{B} . The multiparameter pencil

$$A(z) := z_1 A_1 + z_2 A_2 + \dots + z_n A_n$$

is an important subject of study in various fields, for example in algebraic geometry [38], math physics [2], PDE [3], group theory [21], etc., and more recently in the settlement of the Kadison-Singer Conjecture ([32]). In these studies, the primary interest was in the case when A is a tuple of self-adjoint operators. For general tuples, the notion of projective joint spectrum is defined in [39] and some general study was initiated (cf. [5], [7], [9], [35]).

Definition. For a tuple $A = (A_1, A_2, \dots, A_n)$ of elements in a unital Banach algebra \mathcal{B} , its projective joint spectrum

$$P(A) = \{z \in \mathbb{C}^n \mid A(z) \text{ is not invertible}\}.$$

The *projective resolvent set* refers to the complement $P^c(A) = \mathbb{C}^n \setminus P(A)$.

Various notions of joint spectra for (commuting) tuples have been defined in the past, for instance the Harte spectrum ([24, 25]) and the Taylor spectrum ([16, 36, 37]), and they are important topics in multivariable operator theory. The projective joint spectrum, however, has one notable distinction: it is “base free” in the sense that, instead of using I as a base point and looking at the invertibility of $(A_1 - z_1 I, A_2 - z_2 I, \dots, A_n - z_n I)$ in various constructions, it uses the much simpler pencil $A(z)$. This feature makes the projective joint spectrum computable in

many interesting noncommuting examples, for example the tuple of general compact operators ([35]), the generating tuple for the free group von Neumann algebra ([5]), the Cuntz tuple of isometries (S_1, S_2, \dots, S_n) ([8, 28]) satisfying

$$\begin{aligned} S_i^* S_j &= \delta_{ij} I \quad \text{for } 1 \leq i, j \leq n, \\ \sum_{i=1}^n S_i S_i^* &= I, \end{aligned}$$

where I is the identity, and the tuple $(1, a, t)$ for the infinite dihedral group

$$D_\infty = \langle a, t \mid a^2 = t^2 = 1 \rangle$$

with respect to left regular representation ([22]).

For a tuple A , if the joint resolvent set $P^c(A)$ is nonempty then every path-connected component of $P^c(A)$ is a domain of holomorphy. This fact is a consequence of a result in [40], and it is also proved independently in [29]. On $P^c(A)$, the holomorphic Maurer-Cartan type \mathcal{B} -valued $(1, 0)$ -form

$$\omega_A(z) = A^{-1}(z) dA(z) = \sum_{j=1}^n A^{-1}(z) A_j dz_j$$

played an important role in [39], where it is shown to contain much information about the topology of $P^c(A)$. For example, the de Rham cohomology $H^*(P^c(A), \mathbb{C})$ can be computed by coupling ω_A with invariant multilinear functionals or cyclic cocycles ([9, 39]). Two simple facts are useful here. The first, by differentiating the equation $A(z)A(z)^{-1} = I$, one easily verifies that

$$dA^{-1}(z) = -A^{-1}(z)(dA(z))A^{-1}(z),$$

and consequently,

$$d\omega_A = dA^{-1}(z) \wedge dA(z) = -\omega_A \wedge \omega_A. \quad (0.1)$$

where d is complex differentiation (see Section 1) and \wedge is the wedge product. The second, ω_A is invariant under left multiplication on the tuple A by invertible

elements. To be precise, we let $GL(\mathcal{B})$ denote the set of invertible elements in \mathcal{B} . Then for any $L \in GL(\mathcal{B})$ and $B = LA = (LA_1, LA_2, \dots, LA_n)$, we have for the multiparameter pencils $B(z) = LA(z)$, and hence $P(B) = P(A)$. Moreover,

$$\omega_B = B^{-1}(z)dB(z) = A^{-1}(z)L^{-1}LdA(z) = \omega_A. \quad (0.2)$$

This shows that ω_A is invariant under the left action of $GL(\mathcal{B})$ on the tuple A .

For simplicity, we shall use Einstein convention for summation in many places in this paper. For example, we shall write $\omega_A(z) = A^{-1}(z)A_j dz_j$. When \mathcal{B} is a C^* -algebra, the adjoint of $\omega_A(z)$ is the \mathcal{B} -valued $(0, 1)$ -form

$$\omega_A^*(z) = (A^{-1}(z)A_j)^* d\bar{z}_j.$$

Now we define the *fundamental form* for the tuple A as

$$\Omega_A = -\omega_A^* \wedge \omega_A = (A^{-1}(z)A_j)^* A^{-1}(z)A_k dz_k \wedge d\bar{z}_j. \quad (0.3)$$

Here the negative sign exists because it is conventional to write a $(1, 1)$ -form as linear combinations of $dz_j \wedge d\bar{z}_k$. Often, the factor $\frac{i}{2}$ is used to make $\frac{i}{2}\Omega_A$ a *real* form (cf. (1.3)), but it is not important in this paper.

For a suitable choice of ϕ , such as a faithful state on \mathcal{B} , $\phi(\Omega_A)$ defines a positive definite bilinear form on the holomorphic tangent bundle of $P^c(A)$, thus giving a Hermitian metric on $P^c(A)$. The connection between the metric and the tuple A shall be the primary concern of this paper. Examples indicate that this connection is indeed intimate! A notable feature of this metric is that it has singularities at the joint spectrum $P(A)$. So completeness of the metric is an important issue. For a single operator V , this metric lives on its resolvent set $\rho(V)$ and it adds a new ingredient to classical functional calculus. A particular case is when V has an isolated spectral point, say at 0. In this case, the set of blow up rates at 0 (called *power set*) of the metrics with respect to different choices of ϕ has a somewhat surprising connection with V 's lattice of hyper-invariant subspaces. In particular, when V 's

power set contains more than one points then V has a nontrivial hyperinvariant subspace. A good example is when V is quasi-nilpotent, e.g., $\sigma(V) = \{0\}$, in which case the metric lives on the punctured complex plane. For instance, for the classical Volterra operator the power set is determined to be the interval $(0, 1]$. This paper is organized as follows.

0. Introduction

1. \mathcal{B} -valued differential forms
2. Hermitian metric on $P^c(A)$
3. Completeness
4. Examples about two groups
5. The metric on classical resolvent set
6. Power set at isolated spectral point
7. On hyper-invariant subspaces

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1. \mathcal{B} -VALUED DIFFERENTIAL FORMS

Let M be a complex manifold of dimension n . If $z = (z_1, z_2, \dots, z_n)$ is the coordinate in a local chart, then ∂_i stands for $\frac{\partial}{\partial z_i}$, and $\bar{\partial}_i$ stands for $\frac{\partial}{\partial \bar{z}_i}$. As a convention, we let $\partial = \sum_i \partial_i$, $\bar{\partial} = \sum_i \bar{\partial}_i$, and $d = \partial + \bar{\partial}$. Consider a globally defined smooth $(1, 1)$ -form $\Phi(z) = g_{jk}(z) dz_j \wedge d\bar{z}_k$ expressed in the local chart with the Einstein summation convention. Φ induces a bilinear form $\hat{\Phi}(z) = g_{jk}(z) dz_j \otimes d\bar{z}_k$ on the holomorphic tangent bundle $T(M)$ over the local chart such that

$$\hat{\Phi}(z)(\partial_j, \partial_k) = g_{jk}(z).$$

We say that Φ defines a Hermitian metric on M if the $n \times n$ matrix function $g(z) = (g_{jk}(z))$ is positive definite for each $z \in M$. In this case $g(z)$ is called the associated metric matrix, and $\frac{i}{2}\Phi(z)$ is called the fundamental form of the metric. Here, the constant $\frac{i}{2}$ is to normalize the fundamental form such that

- (a) $\frac{i}{2}\Phi(z)$ is real e.g. $\frac{i}{2}\Phi(z) = \overline{\frac{i}{2}\Phi(z)}$;
- (b) in one variable case $z = x + iy$, we have $\frac{i}{2}dz \wedge d\bar{z} = dx \wedge dy$.

Some geometric concepts are relevant to the study here.

1. A Hermitian metric induced by $\Phi(z)$ is said to be Kähler if $d\Phi = 0$.
2. The Ricci curvature tensor is the $n \times n$ matrix $R(z) = (R_{jk}(z))$, where $R_{jk} = -\partial_j \bar{\partial}_k \log \det g(z)$, and the Ricci form is $Ric_\Phi(z) = R_{jk}(z) dz_j \wedge d\bar{z}_k$.
3. The metric is said to be Ricci flat if $R(z) = 0$ for each $z \in M$.
4. The metric is said to be Einstein if $Ric_\Phi = \lambda \Phi$ for some constant λ , and it is Calabi-Yau if it is Kähler and Ricci flat (e.g. $\lambda = 0$).

One checks that a Hermitian $(1, 1)$ -form $\Phi(z) = g_{ij}(z) dz_i \wedge d\bar{z}_j$ induces a Kähler metric if and only if

$$\partial_k g_{jk}(z) = \partial_j g_{kj}(z), \quad z \in M, \quad (1.1)$$

for all $1 \leq i, j, k \leq n$. Further, Φ is Ricci flat if and only if $\log \det g(z)$ is pluri-harmonic (cf. [34]). Hence in this case $\log \det g(z) = f(z) + \bar{f}(\bar{z})$ holds locally for some holomorphic function f , e.g.

$$\det g(z) = e^{f(z) + \bar{f}(\bar{z})} = |e^{f(z)}|^2. \quad (1.2)$$

Now we consider a smooth \mathcal{B} -valued (p, q) -form $\omega^{p,q}(z)$ defined on a complex domain $R \subset \mathbb{C}^n$, where $0 \leq p, q \leq n$. The set of such forms is denoted by $\omega^{p,q}(R, \mathcal{B})$. Then $\omega^{p,q}(z)$ is said to be ∂ -closed or $\bar{\partial}$ -closed if $\partial\omega^{p,q}(z) = 0$, or respectively $\bar{\partial}\omega^{p,q}(z) = 0$ on R . If $\omega^{p,q}$ is both ∂ -closed and $\bar{\partial}$ closed, then $d\omega^{p,q} = 0$, and in this case we simply say $\omega^{p,q}$ is closed.

For the rest of the paper, unless stated otherwise, we shall assume \mathcal{B} is a C^* -algebra. Recall that for a tuple A , the fundamental form Ω_A is given by (0.3). Now

we check that $\frac{i}{2}\Omega_A$ is *self-adjoint* in the sense that

$$\left(\frac{i}{2}\Omega_A\right)^* = \frac{i}{2}\Omega_A. \quad (1.3)$$

Indeed, one checks by (0.3) that

$$\begin{aligned} \left(\frac{i}{2}\Omega_A\right)^* &= \frac{-i}{2}(-\omega_A^* \wedge \omega_A)^* \\ &= \frac{-i}{2} \left((A^{-1}(z)A_j)^* A^{-1}(z)A_k dz_k \wedge d\bar{z}_j \right)^* \\ &= \frac{-i}{2} (A^{-1}(z)A_k)^* A^{-1}(z)A_j d\bar{z}_k \wedge dz_j \\ &= \frac{-i}{2}(\omega_A^* \wedge \omega_A) \\ &= \frac{i}{2}\Omega_A. \end{aligned}$$

Further, in the case $n = 1$ and A_1 is invertible, one has $A(z) = zA_1$, $P^c(A) = \mathbb{C} \setminus \{0\}$, and $\omega_A(z) = \frac{dz}{z}$. Writing $z = x + yi$ we have

$$\frac{i}{2}\Omega_A = \frac{i(dz \wedge d\bar{z})}{2|z|^2} = \frac{dx \wedge dy}{x^2 + y^2},$$

which is real. For a linear functional $\phi \in \mathcal{B}^*$,

$$\phi\left(\frac{i}{2}\Omega_A\right) = \frac{i}{2}\phi\left((A^{-1}(z)A_j)^* A^{-1}(z)A_k\right) dz_k \wedge d\bar{z}_j,$$

which is a regular $(1, 1)$ -form. Moreover, if ϕ is positive then by (1.3)

$$\overline{\phi\left(\frac{i}{2}\Omega_A\right)} = \phi\left(\frac{-i}{2}\Omega_A^*\right) = \phi\left(\frac{i}{2}\Omega_A\right),$$

and hence $\phi\left(\frac{i}{2}\Omega_A\right)$ is a regular real $(1, 1)$ -form. The following definitions are \mathcal{B} -valued version of Hermitian metric and Kähler metric.

Definition. A smooth \mathcal{B} -valued $(1, 1)$ -form $\Phi(z) = \Phi_{jk}(z)dz_j \wedge d\bar{z}_k$ on a complex domain R is said to be Hermitian if for every $z \in R$ and any set of n complex numbers v_1, v_2, \dots, v_n the double sum $v_i \Phi_{jk}(z) \overline{v_k}$ is a positive element in \mathcal{B} .

Definition. A smooth \mathcal{B} -valued Hermitian $(1, 1)$ -form $\Phi(z)$ is said to be Kähler if it is closed, e.g. $d\Phi(z) = 0$.

Two nuances are worth mentioning. First, there is a difference between a self-adjoint \mathcal{B} -valued $(1, 1)$ -form as defined in (1.3) and a Hermitian \mathcal{B} -valued $(1, 1)$ -form defined above– the latter requires positivity of \mathcal{B} -valued matrix $(\Phi_{ij}(z))$. Second, in the scalar-valued case, for a Hermitian metric defined by $\Omega(z) = g_{jk}(z)dz_j \wedge d\bar{z}_k$, the complex metric matrix $(g_{jk}(z))$ needs to be positive definite, but in the \mathcal{B} -valued case, it is only required that $v_j \Phi_{jk}(z) \bar{v}_k \geq 0$ in \mathcal{B} . For the fundamental form Ω_A and any set of n complex numbers v_1, v_2, \dots, v_n , one checks by (0.3) that the double sum

$$\begin{aligned} v_j A_k^* (A^{-1}(z))^* A^{-1}(z) A_j \bar{v}_k &= \bar{v}_k A_k^* (A^{-1}(z))^* A^{-1}(z) v_j A_j \\ &= (A^{-1}(z) A(v))^* (A^{-1}(z) A(v)) \geq 0. \end{aligned} \quad (1.4)$$

So Ω_A is Hermitian. The tuple $A = (I, A_1, A_2, \dots, A_n)$ is used in some places for normalization purposes. This extension guarantees two convenient facts:

- 1) $P^c(A)$ is not empty, because it contains at least $(1, 0, 0, \dots, 0)$;
- 2) the identity I is in the range of $A(z)$.

Proposition 1.1. *For the tuple $A = (I, A_1, A_2, \dots, A_n)$, the following statements are equivalent.*

- (i) A is commuting.
- (ii) $\omega_A(z)$ is closed.
- (iii) $\Omega_A(z)$ is Kähler.

Proof. Using (0.1) and the fact that $dz_j \wedge dz_k = -dz_k \wedge dz_j$, one checks that

$$\begin{aligned} d\omega_A &= -\omega_A \wedge \omega_A \\ &= A^{-1}(z)A_j A^{-1}(z)A_k dz_j \wedge dz_k \\ &= \sum_{j < k} [A^{-1}(z)A_j, A^{-1}(z)A_k] dz_j \wedge dz_k, \end{aligned} \tag{1.5}$$

where $[a, b] = ab - ba$. So when A is a commuting tuple we have

$$[A^{-1}(z)A_j, A^{-1}(z)A_k] = 0, \quad z \in P^c(A),$$

and consequently $\partial\omega_A = d\omega_A = 0$. This shows (i) implies (ii).

Recall that $\Omega_A(z) = -\omega_A^* \wedge \omega_A$. Since $\omega_A(z)$ is holomorphic

$$\begin{aligned} d\Omega_A &= -(\partial + \bar{\partial})(\omega_A^* \wedge \omega_A) \\ &= -(\partial\omega_A^* + \bar{\partial}\omega_A^*) \wedge \omega_A + \omega_A^* \wedge (\partial\omega_A + \bar{\partial}\omega_A) \\ &= -((\partial\omega_A)^* \wedge \omega_A) + (\omega_A^* \wedge \partial\omega_A). \end{aligned} \tag{1.6}$$

So if $\omega_A(z)$ is closed then $d\Omega_A = -(0 \wedge \omega_A) + (\omega_A^* \wedge 0) = 0$. This shows (ii) implies (iii). Now if Ω_A is Kähler then by (1.6)

$$0 = d\Omega_A = -(\bar{\partial}\omega_A^*) \wedge \omega_A + \omega_A^* \wedge \partial\omega_A. \tag{1.7}$$

Since the first term in (1.6) is a $(1, 2)$ -form and the second is a $(2, 1)$ -form, the sum is 0 only if both terms are 0. Since $\partial\omega_A = d\omega_A$, by (1.5) and (1.7) we have

$$\begin{aligned} 0 &= \omega_A^* \wedge d\omega_A \\ &= (A^{-1}(z)A_i)^* \sum_{j < k} [A^{-1}(z)A_j, A^{-1}(z)A_k] d\bar{z}_i \wedge dz_j \wedge dz_k, \end{aligned}$$

which implies that on $P^c(A)$,

$$(A^{-1}(z)A_i)^* [A^{-1}(z)A_j, A^{-1}(z)A_k] = 0, \quad \forall 0 \leq i, j, k \leq n.$$

Setting $i = 0$ and using the fact $A(z) = A_0 = I$ at $z = (1, 0, 0, \dots, 0)$, we have

$$A_k A_j = A_j A_k, \forall 1 \leq j, k \leq n,$$

e.g. the tuple A is commuting. So (iii) implies (i). \square

In Proposition 1.1, if the tuple is not normalized e.g. I is not in the tuple, then (iii) may not imply (i).

Example 1.2. Consider a pair $A = (A_1, A_2)$, where A_1 is invertible. Then writing $A(z) = A_1(z_1 + z_2 A_1^{-1} A_2)$ and using (0.2), we have

$$\omega_A = (z_1 I + z_2 A_1^{-1} A_2)^{-1} d(z_1 I + z_2 A_1^{-1} A_2).$$

Since I commutes with $A_1^{-1} A_2$, the fundamental form Ω_A is Kähler by Proposition 1.1. But clearly A_1 may not commute with A_2 .

For every $\phi \in B^*$, we have $d\phi(\omega_A) = \phi(d\omega_A)$. The following is then an easy consequence of Propositions 1.1.

Corollary 1.3. Let A be a commuting tuple. Then $\phi(\omega_A) \in H^1(P^c(A), \mathbb{C})$ for every $\phi \in \mathcal{B}^*$.

Proposition 1.1 raises a natural question: whether ω_A is exact e.g. whether there is a \mathcal{B} -valued smooth function $f(z)$ on $P^c(A)$ such that $\omega_A = df$. We address this issue now. For every $\phi \in \mathcal{B}^*$ and a \mathcal{B} -valued smooth (p, q) -form $\omega^{p,q}(z)$ on a complex domain R , the coupling $\phi(\omega^{p,q}(z))$ is a scalar-valued smooth (p, q) -form on R . For example by (0.3),

$$\phi(\Omega_A) = \frac{i}{2} \phi \left(A^{-1}(z) A_j \right)^* A^{-1}(z) A_k dz_k \wedge d\bar{z}_j. \quad (1.8)$$

Moreover, $\omega^{p,q}(z) = 0$ if and only if $\phi(\omega^{p,q}(z)) = 0$ for every $\phi \in \mathcal{B}^*$. The fact

$$\partial\phi(\omega^{p,q}(z)) = \phi(\partial\omega^{p,q}(z))$$

is nicely expressed in the following commuting diagram

$$\begin{array}{ccc}
 \omega^{p,q}(R, \mathcal{B}) & \xrightarrow{\partial_p} & \omega^{p+1,q}(R, \mathcal{B}) \\
 \downarrow \phi & & \downarrow \phi \\
 \omega^{p,q}(R, \mathbb{C}) & \xrightarrow{\partial_p} & \omega^{p+1,q}(R, \mathbb{C}),
 \end{array} \tag{1.9}$$

and the parallel diagram for $\bar{\partial}$. Here the subscript p in ∂_p is to indicate the space on which ∂ acts. Since $d = \partial + \bar{\partial}$, we see that $\omega^{p,q}(z)$ is closed, i.e. $d\omega^{p,q} = 0$, if and only if $\phi(\omega^{p,q}(z))$ is closed for every $\phi \in \mathcal{B}^*$. Since by [29, 40] the joint resolvent set $P^c(A)$ is union of domains of holomorphy, we shall only consider the case R is a domain of holomorphy. In this case, its de Rham cohomology $H^p(R, \mathbb{C})$ (or $H^p(R, \mathcal{B})$) can be computed through holomorphic forms. We denote the set of \mathcal{B} -valued holomorphic p -forms by $\omega^p(R, \mathbb{C})$. Note that in this case $d = \partial$. We refer the readers to [34] for details on domains of holomorphy and holomorphic forms.

Since $\partial_p \partial_{p-1} = 0$, one naturally has \mathcal{B} -valued cohomology groups

$$H^p(R, \mathcal{B}) = \ker \partial_p / \text{im } \partial_{p-1}, \quad p \geq 0, \tag{1.10}$$

where ∂_{-1} is the trivial inclusion map $\{0\} \rightarrow \omega^0(R, \mathcal{B})$. It then follows from the following commuting diagram for holomorphic forms

$$\begin{array}{ccc}
 \omega^p(R, \mathcal{B}) & \xrightarrow{\partial_p} & \omega^{p+1}(R, \mathcal{B}) \\
 \downarrow \phi & & \downarrow \phi \\
 \omega^p(R, \mathbb{C}) & \xrightarrow{\partial_p} & \omega^{p+1}(R, \mathbb{C}),
 \end{array} \tag{1.11}$$

that a form ω is in $H^p(R, \mathcal{B})$ if and only if $\phi(\omega)$ is in $\omega^p(R, \mathbb{C})$ for every $\phi \in \mathcal{B}^*$. It was shown in [39] that if \mathcal{B} is a Banach algebra with a trace ϕ , then for every tuple A such that $P^c(A)$ is nonempty, the 1-form $\phi(\omega_A)$ is a nontrivial element in $H^1(P^c(A), \mathbb{C})$. The above observations thus lead to the following

Corollary 1.4. *Let \mathcal{B} be a unital Banach algebra with a trace and A be a commuting tuple of elements in \mathcal{B} . Then ω_A is a nontrivial element in $H^1(P^c(A), \mathcal{B})$.*

Proof. By Proposition 1.1, we see that ω_A is closed when A is a commuting tuple. If it were exact, then there exists \mathcal{B} -valued smooth function $f(z)$ on $P^c(A)$ such that $\omega_A = df$. If ϕ is the trace, then it follows that

$$\phi(\omega_A) = \phi(df) = d\phi(f).$$

Since $\phi(f)$ is globally defined on $P^c(A)$, the 1-form $\phi(\omega_A)$ is a trivial element in $H^1(P^c(A), \mathbb{C})$ contradicting with the above mentioned fact in [39] that $\phi(\omega_A)$ is a nontrivial element in $H^1(P^c(A), \mathbb{C})$. \square

Proposition 1.1 indicates that the commutativity of tuples A has a natural homological interpretation.

2. HERMITIAN METRIC ON $P^c(A)$

Recall that for a C^* -algebra \mathcal{B} , a smooth \mathcal{B} -valued $(1, 1)$ -form $\Phi(z) = \Phi_{jk}(z)dz_j \wedge d\bar{z}_k$ on a domain R is said to be Hermitian if for any set of n complex numbers v_1, v_2, \dots, v_n the double sum $v_j \Phi_{jk}(z) \overline{v_k}$ is a positive element in \mathcal{B} for every $z \in R$. It follows that if $\phi \in \mathcal{B}^*$ is a positive linear functional, then the matrix $(\phi(\Phi_{jk}(z)))$ is positive semi-definite for every $z \in R$. By (1.4), we see that the principle $(1, 1)$ -form $\Omega_A = -\omega_A^* \wedge \omega_A$ is Hermitian. This section defines some natural Hermitian metrics on $P^c(A)$ through Ω_A and certain states on \mathcal{B} .

A bounded linear functional ϕ on a C^* -algebra \mathcal{B} is called a state if it is positive and $\phi(I) = 1$. Consider a tuple $A = (A_1, A_2, \dots, A_n)$ of elements in \mathcal{B} . Given a state ϕ , then on $P^c(A)$ we compute that

$$\begin{aligned} \phi(\Omega_A(z)) &= -\phi(\omega_A^* \wedge \omega_A) \\ &= \phi(A_k^*(A^{-1}(z))^* A^{-1}(z) A_j) dz_j \wedge d\bar{z}_k \\ &:= g_{jk}(z) dz_j \wedge d\bar{z}_k. \end{aligned} \tag{2.1}$$

As observed in the previous paragraph, the $n \times n$ matrix $g(z) = (g_{jk}(z))$ is positive semi-definite for every $z \in P^c(A)$. The question is for what ϕ the matrix $g(z)$ is positive definite on $P^c(A)$, or in other words, when does $\phi(\Omega_A)$ define a Hermitian metric on $P^c(A)$? To this end, we consider the operator space $\mathcal{H}_A = \text{span}\{A_1, A_2, \dots, A_n\}$. A state ϕ on \mathcal{B} is said to be *faithful on \mathcal{H}_A* if for every nonzero element $h \in \mathcal{H}_A$ we have $\phi(h^*h) > 0$.

Proposition 2.1. *Let ϕ be a state on C^* -algebra \mathcal{B} , then $\phi(\Omega_A)$ defines a Hermitian metric on $P^c(A)$ if and only if ϕ is faithful on \mathcal{H}_A .*

Proof. For every fixed $z \in P^c(A)$, since $A(z) = z_1 A_1 + z_2 A_2 + \dots + z_n A_n$ is invertible, there are numbers $\alpha > \beta > 0$ such that

$$\beta I \leq (A^{-1}(z))^* A^{-1}(z) \leq \alpha I.$$

Note that α and β depend on z , but it is not important here. For every nonzero vector $v = (v_1, v_2, \dots, v_n) \in \mathbb{C}^n$, one checks that

$$\begin{aligned} v_j g_{jk}(z) \bar{v}_k &= v_j \phi(A_k^* (A^{-1}(z))^* A^{-1}(z) A_j) \bar{v}_k \\ &= \phi(\bar{v}_k A_k^* (A^{-1}(z))^* A^{-1}(z) v_j A_j) \\ &= \phi((A^{-1}(z) A(v))^* (A^{-1}(z) A(v))) = \phi(A^*(v) (A^{-1}(z))^* A^{-1}(z) A(v)). \end{aligned}$$

Since

$$\beta A^*(v) A(v) \leq A^*(v) (A^{-1}(z))^* A^{-1}(z) A(v) \leq \alpha A^*(v) A(v)$$

and ϕ is positive, it follows that

$$\beta \phi(A^*(v) A(v)) \leq v_j g_{jk}(z) \bar{v}_k \leq \alpha \phi(A^*(v) A(v)).$$

Hence $(g_{jk}(z))$ is positive definite for every $z \in P^c(A)$ if and only if ϕ is faithful on \mathcal{H}_A . \square

Example 2.2. A state ϕ on a C^* -algebra \mathcal{B} is called a faithful tracial state if $\phi(ab) = \phi(ba)$ for any $a, b \in \mathcal{B}$ and $\phi(a^*a) > 0$ when $a \neq 0$. If ϕ is a faithful tracial state, then it is clearly a faithful state on \mathcal{H}_A . In this case, we shall write the metric matrix (g_{ij}) as $g_A(z)$.

Example 2.3. Assume \mathcal{B} is a C^* -algebra of operators acting on a Hilbert space \mathcal{H} . Let ϕ_x be a vector state defined on \mathcal{B} by $\phi_x(a) = \langle ax, x \rangle$, where $a \in \mathcal{B}$ and x is fixed with $\|x\| = 1$. If $\{A_1x, A_2x, \dots, A_nx\}$ is linearly independent, then for any nonzero vector $(c_1, c_2, \dots, c_n) \in \mathbb{C}^n$, we have

$$\phi_x((c_i A_i)^*(c_i A_i)) = \|c_i A_i x\|^2 > 0,$$

e.g. ϕ is faithful on \mathcal{H}_A . In this case we shall write the metric matrix (g_{ij}) as $g_{A,x}(z)$.

The metrics g_A and $g_{A,x}$, as well as their relations with the tuple A , shall be our main focus for the remaining part of the paper. The following fact follows from Propositions 1.1, 2.1 and the commuting diagram (1.9).

Proposition 2.4. If A is a commuting tuple then $\phi(\Omega_A)$ defines a Kähler metric on $P^c(A)$ for every state ϕ faithful on \mathcal{H}_A .

We end this section by a question motivated by Proposition 2.4.

Problem 1. Find conditions on A and ϕ such that $\phi(\Omega_A)$ defines a Calabi-Yau metric on $P^c(A)$.

In the case of a single operator, Problem 1 is studied in Section 5 and it has an interesting answer (cf. Proposition 5.2).

3. COMPLETENESS

Once a metric is defined on a set, an immediate question is whether the set is complete under the metric. In this section we will study this problem for $P^c(A)$ with respect to the metric defined by g_A and $g_{A,x}$ as in Examples 2.2 and 2.3. Since $\omega_A(z) = A^{-1}(z)dA(z)$ resembles the derivative of logarithmic function, it is natural to expect that the logarithm function shall play an important role here.

In [18], Fuglede and Kadison defined the following notion of determinant

$$\det x = \exp(\phi(\log |x|)),$$

for invertible elements x in a finite Von Neumann algebra \mathcal{B} with a normalized trace ϕ . Here $|x| = \sqrt{x^*x}$ and $\log |x|$ is defined by the functional calculus, e.g.

$$\log |x| = \int_{\sigma(|x|)} \log \lambda dE(\lambda),$$

where $E(\lambda)$ is the associated projection-valued spectral measure. Then

$$\phi(\log |x|) = \int_{\sigma(|x|)} \log \lambda d\phi(E(\lambda)). \quad (3.1)$$

Since x is invertible, the spectrum $\sigma(|x|)$ is bounded below by a positive number. Hence the integral in (3.1) is greater than $-\infty$, which means $\det x \neq 0$. However, there are non-invertible (singular) elements x for which the improper integral in (3.1) is convergent (hence $\det x \neq 0$). This is essentially due to the absolute convergence of

$$\int_0^1 \log t dt.$$

An example of such element is given in [18]. For elements x such that the integral in (3.1) is equal to $-\infty$, $\det x$ is naturally defined to be 0. This extends \det to all elements in \mathcal{B} , and by [18] it is continuous at non-singular elements and upper semi-continuous at singular elements with respect to the norm topology of \mathcal{B} .

FK-determinant has been well-studied in many papers, and we refer readers to [23] for a survey. In particular, it was generalized to C^* -algebras in [27] as follows.

Assume $GL(\mathcal{B})$ is path-connected, $x \in GL(\mathcal{B})$ and $x(t)$ is a piece-wise smooth path in $GL(\mathcal{B})$ such that $x(0) = I$ and $x(1) = x$, then when the integral

$$\int_0^1 \phi(x^{-1}(t)dx(t))$$

is independent of path (which is the case for many C^* -algebras \mathcal{B} !), the following

$$\det_* x = \exp \left(\int_0^1 \phi(x^{-1}(t)dx(t)) \right) \quad (3.2)$$

defines a notion of determinant for invertible elements x . One sees that \det_* may take on complex values, and it is shown in [23] that

$$|\det_* x| = \exp \left(\operatorname{Re} \left(\int_0^1 \phi(x^{-1}(t)dx(t)) \right) \right) = \det x.$$

Clearly, $\det I = 1$, and for a fixed $0 \leq s \leq 1$ we have

$$\det_* x(s) = \exp \left(\int_0^s \phi(x^{-1}(t)dx(t)) \right),$$

so it follows that

$$\phi(x^{-1}(t)dx(t)) = d \log \det_* x(t). \quad (3.3)$$

Here d is the differential with respect to t . This formula will be used quite a few times in what follows. The following definition is needed for our study.

Definition. Consider a C^* -algebra \mathcal{B} with a faithful tracial state ϕ . Then an element x will be called ϕ -singular if its FK-determinant $\det x = 0$. Furthermore, for a tuple A of elements in \mathcal{B} , a point $p \in P(A)$ is said to be ϕ -singular if $A(p)$ is ϕ -singular.

Now we begin to study the completeness of the metric defined by g_A on $P^c(A)$ in Example 2.2. Let $[P^c(A)]$ denote the completion of $P^c(A)$ with respect to the metric g_A . First, consider two points p and q that lie in the same connected component of

$P^c(A)$, and let $\gamma = \{z(t) : 0 \leq t \leq 1\}$ be a piecewise smooth path such that $z(0) = p$ and $z(1) = q$. The length of γ with respect to metric $g_A = (g_{ij})$ is

$$L(\gamma) = \int_0^1 \sqrt{z'_i(t)g_{ij}(z(t))\overline{z'_j(t)}}dt.$$

The distance from p to q with respect to metric g_A is the infimum of $L(\gamma)$ over all such paths, e.g.

$$\text{dist}(p, q) = \inf_{\gamma} L(\gamma).$$

Lemma 3.1. *Let \mathcal{B} be a C^* -algebra with a faithful tracial state ϕ . If p and q are in the same connected component of $P^c(A)$ then*

$$\text{dist}(p, q) \geq | \phi(\log |A(p)|) - \phi(\log |A(q)|) |.$$

Proof. First of all, by (2.1) we have $g_{ij}(z) = \phi((A^{-1}(z)A_j)^*A^{-1}(z)A_i)$. Let p and q be two points in the connected component U of $P^c(A)$, and let $\gamma = \{z(t) | 0 \leq t \leq 1\}$ be a piecewise smooth path such that $z(0) = p$ and $z(1) = q$. Then on γ we have $A'(z(t)) = z'_j(t)A_j$, and the length of γ can be computed as

$$\begin{aligned} L(\gamma) &= \int_0^1 \sqrt{z'_i(t)g_{ij}(z(t))\overline{z'_j(t)}}dt \\ &= \int_0^1 \sqrt{z'_i(t)\phi((A^{-1}(z)A_j)^*A^{-1}(z)A_i)\overline{z'_j(t)}}dt \\ &= \int_0^1 \sqrt{\phi((A^{-1}(z(t))z'_j(t)A_j)^*A^{-1}(z(t))z'_i(t)A_i)}dt \\ &= \int_0^1 \sqrt{\phi((A^{-1}(z(t))A'(z(t)))^*A^{-1}(z(t))A'(z(t)))}dt. \end{aligned} \quad (3.4)$$

Since ϕ is a state, for every $a, b \in \mathcal{B}$ one has $|\phi(ab)|^2 \leq \phi(a^*a)\phi(b^*b)$. In particular, $|\phi(a)|^2 \leq \phi(a^*a)$. Hence

$$\begin{aligned} L(\gamma) &\geq \int_0^1 |\phi(A^{-1}(z(t))A'(z(t)))|dt = \int_0^1 |\phi[A^{-1}(z(t))dA(z(t))]| \\ &= \int_0^1 |\phi(\omega_A(z(t)))| \geq \left| \int_0^1 \phi(\omega_A(z(t))) \right|. \end{aligned}$$

By (3.3) we have

$$\begin{aligned}
L(\gamma) &\geq \left| \int_0^1 d \log \det_* A(z(t)) \right| \\
&= \left| \log \det_* A(p) - \log \det_* A(q) \right| \\
&= \left| \log |\det_* A(p)| - \log |\det_* A(q)| + i(\operatorname{Arg} \det_* A(p) - \operatorname{Arg} \det_* A(q)) \right| \\
&\geq \left| \log |\det_* A(p)| - \log |\det_* A(q)| \right| = \left| \log \det A(p) - \log \det A(q) \right|. \quad (3.5)
\end{aligned}$$

Since γ is arbitrary, we have

$$\begin{aligned}
\operatorname{dist}(p, q) &= \inf_{\gamma} L(\gamma) \geq \left| \log \det A(p) - \log \det A(q) \right| \\
&= \left| \phi(\log |A(p)|) - \phi(\log |A(q)|) \right|. \quad (3.6)
\end{aligned}$$

□

The proof of Lemma 3.1 can be modified a little to accommodate the case in which q is a boundary point of a path-connected component of $P^c(A)$. First, we observe that in this case $q \in P(A)$. Suppose there exists a piece-wise smooth path $z(t)$ such that $z(t) \in P^c(A)$, $0 \leq t < 1$ but $q = z(1) \in P(A)$. Then by the proof above we have

$$\begin{aligned}
L(\gamma) &\geq \lim_{s \rightarrow 1^-} \int_0^s |\phi(\omega_A(z(t)))| \\
&\geq \limsup_{s \rightarrow 1^-} \left| \log \det A(p) - \log \det A(z(s)) \right|. \quad (3.7)
\end{aligned}$$

If q is a ϕ -singular point, we have $\det A(z(1)) = \det A(q) = 0$. Note that 0 is the minimum possible value for \det . Hence the upper semi-continuity of $\det A(z(s))$ implies

$$\lim_{s \rightarrow 1} \det A(z(s)) = 0.$$

Therefore by (3.7) we have $L(\gamma) = \infty$ for every such path γ in $P^c(A)$ connecting $p \in P^c(A)$ to $q \in P(A)$, and consequently

$$\operatorname{dist}(p, q) = \inf_{\gamma} L(\gamma) = \infty.$$

Since q has infinite distance to every $p \in P^c(A)$, we have the following

Corollary 3.2. *For a tuple A in a C^* -algebra \mathcal{B} with a faithful tracial state ϕ , if $q \in \partial P^c(A)$ is ϕ -singular, then $q \notin [P^c(A)]$.*

If \mathcal{B} is a matrix subalgebra in $M_k(\mathbb{C})$, we let Tr and det stand for the ordinary trace and respectively determinant on $k \times k$ matrices. Then for any n -tuple A , the joint spectrum $P(A)$ is the hypersurface $\{det A(z) = 0\}$ in \mathbb{C}^n . Clearly, $\partial P^c(A) = P(A)$. Let $\phi = \frac{1}{k}Tr$. Then ϕ is a tracial state on $M_k(\mathbb{C})$, and in this case the FK-determinant and the usual determinant satisfies the relation $det x = |det x|^{1/k}$ (cf. [18]). Clearly, in this case every point in $P(A)$ is ϕ -singular. Since the factor $1/k$ is not important, thus we have the following

Corollary 3.3. *For every tuple A of $k \times k$ matrices, the metric on $P^c(A)$ defined by $Tr(\Omega_A)$ is complete.*

4. EXAMPLE ON TWO GROUPS

The simplest case is when \mathcal{B} is the complex plane and $A = 1$. Then obviously $A(z) = z$, and $P^c(A)$ is the punctured complex plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Clearly, $\omega_A(z) = \frac{dz}{z}$, and the metric is then given by $\Omega_A = \frac{1}{|z|^2} dz \wedge d\bar{z}$. Observe that in this case the metric is Ricci flat and is Kähler for trivial reasons, e.g., it is Calabi-Yau. Moreover, one checks easily that $\frac{1}{|z|^2} dz \wedge d\bar{z}$ is invariant under the action of \mathbb{C}^\times by multiplication $z \rightarrow \alpha z$, $\alpha \in \mathbb{C}^\times$, and is also invariant under reflection $z \rightarrow 1/\bar{z}$. Some of these facts can be generalized to higher dimensions.

Corollary 4.1. *Let A_1, A_2, \dots, A_n be a basis for the matrix algebra $M_k(\mathbb{C})$, where of course $n = k^2$. Then $Tr(\Omega_A)$ defines a complete, left invariant and Ricci flat metric on the complex general linear group $GL(k)$.*

Proof. First, since A_1, A_2, \dots, A_n is a basis for the matrix algebra $M_k(\mathbb{C})$, every matrix is of the form $A(z)$ for some z . Hence $z \in P^c(A)$ if and only if

$A(z) \in GL(k)$. And this provides an identification of $P^c(A)$ with $GL(k)$. So in particular $Tr(\Omega_A)$ defines a metric on $GL(k)$. Recall that

$$Tr(\Omega_A) = -Tr(\omega_A^* \wedge \omega_A) = Tr((A^{-1}(z)A_j)^* A^{-1}(z)A_i) dz_i \wedge d\bar{z}_j.$$

For $1 \leq i \leq n$, we write $A_i = (a_1^i, a_2^i, \dots, a_k^i)$, where a_m^i , $1 \leq m \leq k$, is the m th column of A_i . Then

$$\begin{aligned} & (A^{-1}(z)A_j)^* A^{-1}(z)A_i \\ &= (A^{-1}(z)a_1^j, A^{-1}(z)a_2^j, \dots, A^{-1}(z)a_k^j)^* (A^{-1}(z)a_1^i, A^{-1}(z)a_2^i, \dots, A^{-1}(z)a_k^i), \end{aligned}$$

and hence

$$Tr((A^{-1}(z)A_j)^* A^{-1}(z)A_i) = \sum_m \langle A^{-1}(z)a_m^i, A^{-1}(z)a_m^j \rangle_{\mathbb{C}^k}. \quad (4.1)$$

Let α be the $n \times n$ -matrix $(a_m^i)_{1 \leq m \leq k, 1 \leq i \leq n}$, whose i -th column (in \mathbb{C}^n) is denoted by α_i , and let I_k be the $k \times k$ identity matrix. Then $A^{-1}(z) \otimes I_k$ is the $n \times n$ block matrix with $A^{-1}(z)$ on the diagonal. So we can write the sum in (4.1) as

$$\langle (A^{-1}(z) \otimes I_k)\alpha_i, (A^{-1}(z) \otimes I_k)\alpha_j \rangle_{\mathbb{C}^n}. \quad (4.2)$$

Since A_1, A_2, \dots, A_n are linearly independent, α is invertible, and one checks by direct computation that the metric matrix

$$\begin{aligned} g(z) &= (Tr((A^{-1}(z)A_j)^* A^{-1}(z)A_i))_{n \times n} \\ &= (\langle (A^{-1}(z) \otimes I_k)\alpha_i, (A^{-1}(z) \otimes I_k)\alpha_j \rangle)_{n \times n} \\ &= \alpha^* (A^{-1}(z) \otimes I_k)^* (A^{-1}(z) \otimes I_k) \alpha. \end{aligned}$$

It follows that $\det g(z) = |\det \alpha|^2 |\det A^{-k}(z)|^2$. Since $\det A^{-k}(z)$ is holomorphic on $P^c(A)$, the metric defined by $Tr \Omega_A$ is Ricci flat by the remarks after (1.1).

Moreover, because A_1, A_2, \dots, A_n is a basis for $M_k(\mathbb{C})$, for every fixed $x \in GL(k)$, $xA(z) = A(w)$, where $w \in P^c(A)$ and is uniquely determined by z . This gives rise to a natural action L of $GL(k)$ on $P^c(A)$ given by $L_x(z) = w$. Since

by (0.2), ω_A is invariant under left action of $GL(k)$, the metric defined by $Tr\Omega_A$ is invariant under the action of L . The completeness of the metric follows from Corollary 3.3. \square

Now we turn to the infinite dihedral group

$$D_\infty = \langle a, t \mid a^2 = t^2 = e \rangle.$$

Clearly, D_∞ is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. So in some sense it is the simplest non-abelian group. But D_∞ is also highly non-trivial because some complicated groups can be constructed through the actions of D_∞ ([33]).

Let λ be the left regular representation of D_∞ on $l^2(D_\infty)$ such that

$$\lambda(g)\delta_x = \delta_{gx}, \quad \forall g, x \in D_\infty,$$

where δ_x is the characteristic function of the singleton set $\{x\}$. Then the group von Neuman algebra generated by $\lambda(a)$ and $\lambda(t)$ (denoted by \mathcal{B}_D) is a type II_1 factor with a faithful tracial state tr defined by

$$tr(h) = \langle h\delta_e, \delta_e \rangle, \quad h \in \mathcal{B}_D.$$

D_∞ contains the normal subgroup $H = \langle at \rangle$. It is easy to see that $D_\infty/H = \mathbb{Z}_2$. So elements in D_∞ is of the form $(at)^k$ or $t(at)^k$, where $k \in \mathbb{Z}$. The Hilbert space $l^2(D_\infty)$ then can be written as $L \oplus tL$, where $L = l^2(H)$. It was shown in [22] that, up to unitary equivalence, we have the following decomposition:

$$\lambda(a) = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}, \quad \lambda(t) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (4.3)$$

where $(Tf)(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$ is the bilateral shift operator on $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$.

For the tuple $R = (I, \lambda(a), \lambda(t))$, we let $R_*(z) = I + z_1\lambda(a) + z_2\lambda(t)$ and $P(R_*)$ be the set of $z \in \mathbb{C}^2$ such that $R_*(z)$ is not invertible in \mathcal{B}_D . Then by [22]

we have

$$P(R_*) = \bigcup_{-1 \leq x \leq 1} \{z \in \mathbb{C}^3 : 1 - z_1^2 - z_2^2 - 2z_1 z_2 x = 0\}, \quad (4.4)$$

and the Fuglede-Kadison determinant of $R_*(z)$ is

$$\det R_*(z) = \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \log |1 - z_1^2 - z_2^2 - 2z_1 z_2 \cos \theta| d\theta\right), \quad z \in \mathbb{C}^2. \quad (4.5)$$

It is determined in [22] (through (4.5)) that the set of tr -singular points in $P(R_*)$ is

$$S := \{(\pm 1, 0), (0, \pm 1)\}.$$

We let $\omega_R(z) = R_*^{-1}(z) dR_*(z)$ and as before set $\Omega_R(z) = -\omega_R^*(z) \wedge \omega_R(z)$. Since tr is faithful, by Proposition 2.1 the $(1, 1)$ -form $tr \Omega_R(z)$ induces a Hermitian metric on $P^c(R_*)$. By Corollary 3.2 the set S is not in the completion of $P^c(R_*)$ with respect to the metric. However, it is not clear if $[P^c(R_*)] = \mathbb{C}^2 \setminus S$.

5. THE METRIC ON CLASSICAL RESOLVENT SET

The remaining part of the paper shall be devoted to the single operator case. In this case, the metrics g_A and $g_{A,x}$ are defined on the classical resolvent set $\rho(A)$. Indeed, it is quite interesting to see what the metrics may reveal about the operator A .

Consider a densely defined (possibly unbounded) linear operator A acting on a Hilbert space \mathcal{H} . A complex number λ is said to be regular for A if $A - \lambda$ has a bounded inverse. The set of regular points is denoted by $\rho(A)$ and is called the resolvent set for A . The spectrum is $\sigma(A) := \mathbb{C} \setminus \rho(A)$. It is well known that $\rho(A)$ is an open set. Let $\omega_A = (A - zI)^{-1} dz$ and ϕ_x be a vector state in Example 2.3, then on the resolvent set $\rho(A)$,

$$\phi_x(\Omega_A) = -\phi_x(\omega_A^* \wedge \omega_A) = \|(A - zI)^{-1} x\|^2 dz \wedge d\bar{z}. \quad (5.1)$$

Since $g_{A,x}(z) = \|(A - zI)^{-1}x\|^2 > 0$ everywhere on $\rho(A)$, it defines a Hermitian metric on $\rho(A)$. When there is no confusion about A , we shall simply write $g_{A,x}$ as g_x .

On the other hand, if A is an element in a C^* -algebra \mathcal{B} with a faithful tracial state ϕ , then

$$\phi(\Omega_A) = -\phi(\omega_A^* \wedge \omega_A) = \phi((A^* - \bar{z}I)^{-1}(A - zI)^{-1})dz \wedge d\bar{z}. \quad (5.2)$$

So $g_A(z) = \phi((A^* - \bar{z}I)^{-1}(A - zI)^{-1})$ in this case. In contrast with the usual Euclidean metric on the complex plane, the two metrics g_x and g_A are dependent on A , the vector x or the trace ϕ , and they may have singularities at spectral points. So the geometry of $\rho(A)$ with respect to the two metrics will reflect properties of A , as well as its interplays with x or ϕ . This and the next section will focus on these two metrics. We begin with a simple example.

Example 5.1. Consider $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ with orthonormal basis $\{e^{in\theta} : n \in \mathbb{Z}\}$, and differential operator

$$Df(\theta) = \frac{df(\theta)}{id\theta}, \quad f \in C^1 \subset L^2,$$

where C^1 is the dense subspace of differentiable functions in L^2 . Then it is well-known that D has eigenvalues n with corresponding eigen-functions $e^{in\theta}$, $n \in \mathbb{Z}$. Therefore,

$$(D - zI)^{-1}e^{in\theta} = \frac{1}{n - z}e^{in\theta},$$

which implies that $\|(D - zI)^{-1}\| < \infty$ for every fixed $z \notin \sigma(D) = \mathbb{Z}$. Moreover, it follows that

$$(D^* - \bar{z}I)^{-1}(D - zI)^{-1}e^{in\theta} = \frac{1}{|z - n|^2}e^{in\theta}, \quad z \in \mathbb{C} \setminus \mathbb{Z}, \quad n \in \mathbb{Z}.$$

This shows that $(D^* - \bar{z}I)^{-1}(D - zI)^{-1}$ is in fact trace class for every $z \in \mathbb{C} \setminus \mathbb{Z}$, and hence by (5.2)

$$\begin{aligned} \text{tr} \Omega_D &= \text{tr}[(D^* - \bar{z}I)^{-1}(D - zI)^{-1}] dz \wedge d\bar{z} \\ &= \left(\sum_{n=-\infty}^{\infty} \frac{1}{|z - n|^2} \right) dz \wedge d\bar{z} \end{aligned}$$

defines a metric on $\mathbb{C} \setminus \mathbb{Z}$ that has poles at integer numbers.

The metric defined by (5.1) gives rise to an interesting and somewhat unexpected connection between geometry and operator theory.

Proposition 5.2. *Assume a densely defined linear operator A on a Hilbert space \mathcal{H} has nonempty resolvent set $\rho(A)$, and x is a unit vector in \mathcal{H} . Then the metric g_x has non-positive Ricci curvature $R(z)$. Moreover, g_x is Ricci flat if and only if x is an eigenvector of A .*

Proof. In this case, the Ricci curvature is a scalar form, and one computes that

$$\begin{aligned} R(z) &= -\partial\bar{\partial} \log \|(A - z)^{-1}x\|^2 \\ &= -\partial \frac{\langle (A - z)^{-1}x, (A - z)^{-2}x \rangle}{\|(A - zI)^{-1}x\|^2} \\ &= -\frac{\|(A - z)^{-2}x\|^2 \|(A - z)^{-1}x\|^2 - |\langle (A - z)^{-2}x, (A - z)^{-1}x \rangle|^2}{\|(A - zI)^{-1}x\|^4}. \end{aligned}$$

So by Cauchy-Schwarz inequality, $R(z) \leq 0$ for all $z \in \rho(A)$, and $R(z)$ vanishes at some point z_0 if and only if $(A - z_0)^{-1}x = \lambda(A - z_0)^{-2}x$, or equivalently $(A - z_0)x = \lambda x$ for some scalar λ (dependent on z_0). So x is an eigenvector of A with eigenvalue $z_0 + \lambda$. \square

Since by (5.1) $\phi_x(\Omega_A)$ is Kähler for trivial reasons, Proposition 5.2 indicates that g_x is Calabi-Yau if and only if x is an eigenvector of A ! This is a partial answer to Problem 1 in Section 2.

In the case \mathcal{B} is a C^* algebra with a faithful tracial state ϕ , the inner product

$$\langle A, B \rangle := \phi(A^*B), \quad A, B \in \mathcal{B},$$

is well-defined, and it induces the norm $\|A\|_\phi = \sqrt{\phi(A^*A)}$ on \mathcal{B} . Then

$$\phi(\Omega_A) = \langle (A - zI)^{-1}, (A - zI)^{-1} \rangle dz \wedge d\bar{z} = \|(A - zI)^{-1}\|_\phi^2 dz \wedge d\bar{z}.$$

Similar to the proof of Proposition 5.2, the Ricci curvature

$$\begin{aligned} R(z) &= -\partial\bar{\partial} \log \|(A - z)^{-1}\|_\phi^2 \\ &= -\partial \frac{\langle (A - z)^{-1}, (A - z)^{-2} \rangle}{\|(A - zI)^{-1}\|_\phi^2} \\ &= -\frac{\|(A - z)^{-2}\|_\phi^2 \|(A - z)^{-1}\|_\phi^2 - |\langle (A - z)^{-2}, (A - z)^{-1} \rangle|^2}{\|(A - zI)^{-1}\|_\phi^4}. \end{aligned}$$

So similarly, by Cauchy-Schwarz inequality $R(z) \leq 0$ on $\rho(A)$ and $R(z_0) = 0$ for some $z_0 \in \rho(A)$ if and only if

$$(A - z_0)^{-1} = \lambda(A - z_0)^{-2}$$

for some scalar λ (dependent on z_0), which is true if and only if $A = (\lambda + z_0)I$.

Thus we have the following

Proposition 5.3. *Suppose \mathcal{B} is a unital C^* algebra with a faithful tracial state ϕ . Then $g_A = \phi((A^* - \bar{z})^{-1}(A - z)^{-1})$ is Ricci flat if and only if A is a scalar multiple of the unit I .*

We use 2×2 matrices to illustrate the above fact.

Example 5.4. *Let $A = (a_{jk})$ be an upper triangular 2×2 matrix and assume ϕ is the normalized trace, e.g.*

$$\phi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \frac{1}{2}(a_{11} + a_{22}).$$

Note that $\phi(I) = 1$. Then by a simple computation, we have

$$g(z) = \frac{1}{|z - a_{11}|^2} + \frac{1}{|z - a_{22}|^2} + \frac{a_{12}}{|z - a_{11}|^2 |z - a_{22}|^2},$$

and one verifies that

$$R(z) = -\frac{|a_{11} - a_{22}|^2 + 2|a_{12}|^2}{(|z - a_{11}|^2 + |z - a_{22}|^2 + |a_{12}|^2)^2}.$$

So in particular, $R(z)$ is constant 0 if and only if A is a scalar multiple of the identity matrix.

In general the metric defined in (5.1) is not complete. For instance, if A has an eigenvalue λ with corresponding unit eigenvector x , then $(A - zI)x = (\lambda - z)x$ or equivalently

$$(A - zI)^{-1}x = (\lambda - z)^{-1}x, \quad z \in \rho(A).$$

Therefore,

$$\phi_x(\Omega_A) = \|(A - zI)^{-1}x\|^2 dz \wedge d\bar{z} = \frac{dz \wedge d\bar{z}}{|\lambda - z|^2}.$$

So in a neighborhood of any other spectral point λ' this metric is equivalent to the Euclidean metric, which is not complete on $\rho(A)$. However, if x instead is a cyclic vector, then the metric may be complete, at least it is so in matrix case.

Proposition 5.5. *If A is a $k \times k$ matrix and x is cyclic for A , then on $\rho(A)$ the metric defined by $g_x = \|(A - zI)^{-1}x\|^2$ is equivalent to that defined by*

$$g_A = \frac{1}{k} \text{Tr}[(A - zI)^{-1} (A - zI)^{-1*}].$$

Proof. It is clear that

$$\begin{aligned} \|(A - zI)^{-1}x\|^2 &\leq \|((A - zI)^{-1})^* (A - zI)^{-1}\| \\ &\leq \text{Tr}[(A - zI)^{-1} (A - zI)^{-1*}]. \end{aligned}$$

For the other direction, since x is cyclic, $\{x, Ax, \dots, A^{k-1}x\}$ form a basis for \mathbb{C}^k . If $\{e_1, e_2, \dots, e_k\}$ is the standard orthonormal basis for \mathbb{C}^k , then the map

$Se_j = A^{j-1}x$, $1 \leq j \leq k$ defines an invertible matrix S . Then one easily checks that

$$\begin{aligned} & Tr[S^*((A - zI)^{-1})^*(A - zI)^{-1}S] \\ &= \sum_{j=1}^k \langle (S^*((A - zI)^{-1})^*(A - zI)^{-1}Se_j, e_j \rangle \\ &= \sum_{j=1}^k \|(A - zI)^{-1}A^{j-1}x\|^2 \leq \left(\sum_{j=1}^k \|A^{j-1}\|^2\right) \|(A - zI)^{-1}x\|^2. \end{aligned}$$

Using the trace-norm inequality $\|XTY\|_1 \leq \|X\|\|Y\|\|T\|_1$, one has

$$\begin{aligned} Tr[((A - zI)^{-1})^*(A - zI)^{-1}] &= Tr[(S^*)^{-1}S^*((A - zI)^{-1})^*(A - zI)^{-1}SS^{-1}] \\ &\leq Tr[S^*((A - zI)^{-1})^*(A - zI)^{-1}S]\|S^{-1}\|^{-2} \\ &\leq \|S^{-1}\|^{-2} \left(\sum_{j=0}^{k-1} \|A^j\|^2\right) \|(A - zI)^{-1}x\|^2. \end{aligned}$$

Setting $\beta = \|S^{-1}\|^{-2}(\sum_{j=0}^{k-1} \|A^j\|^2)^{-1}$, we concludes that

$$\frac{1}{\beta}g_A(z) \leq g_x(z) \leq kg_A(z), \quad z \in \rho(A),$$

e.g. g_A and g_x are equivalent. □

In view of Corollary 3.3, Proposition 5.5 implies that for a square matrix A the metric defined by $g_{A,x}$ on $\rho(A)$ is complete when x is cyclic. This may not be true in infinite dimensional cases. The following is an example.

Example 5.6. Let $H = H^2(\mathbb{D})$ be the classical Hardy space and A be the unilateral shift T_w . It is well-known that $\sigma(T_w) = \overline{\mathbb{D}}$, and that constant 1 is a cyclic vector for T_w . It is an easy computation to verify that

$$g_1(z) = \|(T_w - z)^{-1}1\|^2 = \frac{1}{|z|^2 - 1}, \quad |z| > 1. \quad (5.3)$$

Clearly, the metric is radial, and it blows up at the unit circle. However, it does not mean the metric is complete. Consider the line segment $\gamma(t) = 2 - t$, $0 \leq t \leq 1$. Its length with respect to the metric defined in (5.3) is

$$\begin{aligned} L(\gamma) &= \int_0^1 \frac{|\gamma'(t)|dt}{\sqrt{|\gamma(t)|^2 - 1}} \\ &= \int_0^1 \frac{dt}{\sqrt{(1-t)(3-t)}} \leq \frac{1}{\sqrt{2}} \int_0^1 \frac{dt}{\sqrt{1-t}} = \sqrt{2}. \end{aligned}$$

The path γ is the interval $[1, 2]$, and the computation above shows that its length under the metric in (5.3) is $\sqrt{2}$. This shows that 1 is a limit point of $\rho(T_w)$ under this metric, e.g. $1 \in [\rho(T_w)] \setminus \rho(T_w)$, which indicates that the metric is not complete. Further, by radial symmetry we thus have $\mathbb{T} \subset [\rho(T_w)]$. Furthermore, since g_1 is bounded in the neighborhood of ∞ , it is dominated by the Euclidean metric (up to a scalar multiple). This implies that $\infty \notin [\rho(T_w)]$. In summary, we have $[\rho(T_w)] = \mathbb{C} \setminus \mathbb{D}$, just like that under the Euclidean metric. But of course, the difference is significant as well: $\frac{1}{|z|^2-1}$ blows up at the unit circle, and it is not Ricci flat on $\rho(T_w)$.

Let $\gamma = \{z(t) : 0 \leq t \leq 1\}$ be a positively oriented piecewise smooth closed path in $\rho(A)$ that encloses $\sigma(A)$. The winding number of γ around $\sigma(A)$ is denoted by $n(\gamma)$. The following theorem indicates that the length of γ has a lower bound, no matter how small the spectrum is.

Proposition 5.7. *Assume a densely defined linear operator A has bounded resolvent, and x is a unit vector in \mathcal{H} . Let γ be a closed piece-wise smooth path in $\rho(A)$ that encloses the spectrum $\sigma(A)$, then with respect to the metric $g_{A,x}$ we have $L(\gamma) \geq 2\pi n(\gamma)$.*

Proof. Let Γ be a closed piece-wise smooth path in $\rho(A)$ that encloses $\sigma(A)$ and is enclosed by γ . In other words, the path Γ sits between γ and $\sigma(A)$, and $\Gamma \cap \gamma$ is

empty. Then by functional calculus

$$I = \frac{1}{2\pi i} \int_{\Gamma} (A - \lambda I)^{-1} d\lambda, \quad (5.4)$$

and

$$(A - zI)^{-1} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - z)^{-1} (A - \lambda I)^{-1} d\lambda.$$

Hence

$$\begin{aligned} L(\gamma) &= \int_0^1 |z'(t)| \| (A - z(t)I)^{-1} x \| dt \\ &\geq \left\| \int_0^1 z'(t) (A - z(t)I)^{-1} x dt \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\Gamma} \left(\int_0^1 z'(t) (\lambda - z(t))^{-1} dt \right) (A - \lambda I)^{-1} x d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\Gamma} \left(\int_0^1 d \log(\lambda - z(t)) \right) (A - \lambda I)^{-1} x d\lambda \right\| \end{aligned}$$

Clearly, the inside integral with respect to t is $2\pi i n(\gamma, \lambda)$, where $n(\gamma, \lambda)$ is the winding number of γ around λ . Since Γ is connected and γ properly encloses Γ , the number $n(\gamma, \lambda)$ is independent of λ and is indeed equal to the winding number of γ around $\sigma(A)$. Therefore by (5.4),

$$\begin{aligned} L(\gamma) &\geq 2\pi n(\gamma) \left\| \frac{1}{2\pi i} \int_{\Gamma} (A - \lambda I)^{-1} x d\lambda \right\| \\ &= 2\pi n(\gamma) \|x\| = 2\pi n(\gamma). \end{aligned}$$

□

Several consequences are notable.

(1) If $\sigma(A)$ is not path-connected, and γ only encloses part of σ , say E , then the integral in (5.4) is a projection P . Therefore the last integral in the above proof will give Px , and hence we will have

$$L(\gamma) \geq 2\pi n(\gamma) \|Px\|. \quad (5.5)$$

(2) Assume A has an isolated spectral point, say λ_0 , and γ only encloses $\{\lambda_0\}$. Suppose $Px \neq 0$, then by (5.5), we have $L(\gamma) \geq 2\pi\|Px\|$ no matter how close γ is to λ_0 in the Euclidean sense. This phenomenon occurs because the metric $g_{A,x}$ blows up at λ_0 . In fact, $L(\gamma)$ blows up as γ is getting close to λ_0 , and the blow up rate will be investigated in the next section.

Whether an isolated spectral point λ_0 belongs in $[\rho(A)]$ is a rather subtle issue. In the remarks above Proposition 5.5, we see that if x is an eigenvector of A corresponding to eigenvalue λ then $g_x = \frac{1}{|\lambda - z|^2}$. So if λ' is an isolated spectral point and $\lambda' \neq \lambda$, then $\lambda' \in [\rho(A)]$, which indicates that g_x is not complete on $\rho(A)$. The next result tells about the opposite situation.

Proposition 5.8. *Assume A has an isolated spectral point λ_0 . If there exists $\eta > 0$ and a neighborhood U of λ_0 such that $|z - \lambda_0| \|(A - z)^{-1}x\| \geq \eta$, $\forall z \in U$, then $\lambda_0 \notin [\rho(A)]$.*

Proof. Consider a point $z_0 \in \rho(A)$ and let $\gamma = \{z(t), 0 \leq t \leq 1\}$ be a piecewise smooth path such that $z(0) = z_0$, $z(t) \in \rho(A)$ for $0 \leq t < 1$ and $z(1) = \lambda_0$. Let $a > 0$ be such that $z(t) \in U$ for all $a \leq t \leq 1$. Then with respect to the metric $g_{A,x}$, the length

$$\begin{aligned}
L(\gamma) &= \lim_{b \rightarrow 1^-} \int_0^b |z'(t)| \|(A - z(t)I)^{-1}x\| dt \\
&\geq \lim_{b \rightarrow 1^-} \int_a^b |z'(t)| \|(A - z(t)I)^{-1}x\| dt \\
&\geq \lim_{b \rightarrow 1^-} \int_a^b \frac{\eta |z'(t)|}{|z(t) - \lambda_0|} dt \\
&\geq \eta \lim_{b \rightarrow 1^-} \int_a^b |d \log(z(t) - \lambda_0)| \\
&\geq \eta \lim_{b \rightarrow 1^-} |\log(z(b) - \lambda_0) - \log(z(a) - \lambda_0)| = \infty.
\end{aligned}$$

Since γ is arbitrary, this shows $\text{dist}_g(z_0, \lambda_0) = \infty$ for every $z_0 \in \rho(A)$, and hence λ_0 is not in $[\rho(A)]$. \square

If A is a bounded normal operator then by its spectral resolution we have

$$(A - zI)^{-1} = \int_{\sigma(A)} \frac{1}{\lambda - z} dE(\lambda).$$

Hence

$$\begin{aligned} \|(A - zI)^{-1}x\|^2 &= \langle ((A - zI)^{-1})^*(A - zI)^{-1}x, x \rangle \\ &= \int_{\sigma(A)} \frac{1}{|\lambda - z|^2} d\langle E(\lambda)x, x \rangle. \end{aligned}$$

Let $\gamma = \{z(t) : 0 \leq t \leq 1\}$ be a piecewise smooth path as in the proof of Proposition 5.8. Then similar to the proof there we have

$$\begin{aligned} L(\gamma) &= \int_0^1 |z'(t)| \left(\int_{\sigma(A)} \frac{1}{|\lambda - z(t)|^2} d\langle E(\lambda)x, x \rangle \right)^{1/2} dt \\ &\geq \int_0^1 \left(\int_{\sigma(A)} \frac{|z'(t)|}{|\lambda - z(t)|} d\langle E(\lambda)x, x \rangle \right) dt \\ &= \int_{\sigma(A)} \left(\int_0^1 \left| \frac{d}{dt} \log(\lambda - z(t)) \right| dt \right) d\langle E(\lambda)x, x \rangle \\ &\geq \int_{\sigma(A)} \left| \int_0^1 \frac{d}{dt} \log(\lambda - z(t)) dt \right| d\langle E(\lambda)x, x \rangle \\ &= \int_{\sigma(A)} |\log(\lambda - z(1)) - \log(\lambda - z(0))| d\langle E(\lambda)x, x \rangle. \end{aligned} \quad (5.6)$$

Since $z(0) \in \rho(A)$, the integral of $|\log(\lambda - z(0))|$ is finite, so the convergence of above integral depends on the integral of $|\log(\lambda - \lambda_0)|$, or equivalently, $\log |\lambda - \lambda_0|$. We recall that given a probability measure μ on a compact set $K \in \mathbb{C}$, the logarithmic potential at $z \in \mathbb{C}$ is defined as

$$V_K(z) = \int_K \log \frac{1}{|\lambda - z|} d\mu.$$

Therefore, by (5.6) the length $L(\gamma) = \infty$ if the measure $\langle E(\lambda)x, x \rangle$ on $\sigma(A)$ has infinite logarithmic potential at $z(1) = \lambda_0 \in \sigma(A)$. We state the observations above as

Proposition 5.9. *Let A be a bounded normal operator and $\lambda_0 \in \partial\rho(A)$. If λ_0 is in $[\rho(A)]$ then it has finite logarithmic potential with respect to the measure $\langle E(\lambda)x, x \rangle$ on $\sigma(A)$.*

Example 5.10. *Now we look at a particular case of Proposition 5.9. We consider the bilateral shift T defined on $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ by $Tf(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$. In this case $\sigma(T) = \mathbb{T}$. Similar to Example 5.6 we let $x = 1$ and easily compute that*

$$g_1(z) = \|(T - zI)^{-1}1\| = \frac{1}{\sqrt{|1 - |z|^2|}}, \quad z \notin \mathbb{T}.$$

And as in Example 5.6, the length of the line segment $\gamma(t) = 2 - t$, $0 \leq t \leq 1$ with respect to the metric g_1 satisfies

$$L(\gamma) \leq \sqrt{2}.$$

This shows $1 \in [\rho(T)]$. Since the metric g_1 is radial, we see that $\mathbb{T} \subset [\rho(T)]$. And likewise, we have $\infty \notin [\rho(T)]$, therefore in this case $[\rho(T)] = \mathbb{C}$.

Moreover, in this case $E(e^{i\theta})$ is the orthogonal projection from L^2 to $\chi_{[0, \theta]}L^2$, hence

$$d\langle E(e^{i\theta})1, 1 \rangle = \frac{d\theta}{2\pi}.$$

So the logarithmic potential at $z = 1$ is given by

$$\begin{aligned} V_{\mathbb{T}}(1) &= \int_0^{2\pi} \log \frac{1}{|e^{i\theta} - 1|} \frac{d\theta}{2\pi} = \int_0^{2\pi} \frac{1}{2} \log \frac{1}{4 \sin^2 \frac{\theta}{2}} \frac{d\theta}{2\pi} \\ &= - \int_0^{2\pi} \log 2 + \log \sin \frac{\theta}{2} \frac{d\theta}{2\pi} \\ &= -\log 2 - \int_0^{\pi} \log \sin \theta \frac{d\theta}{\pi}. \end{aligned} \tag{5.7}$$

Using substitution and the following known fact about logarithmic integral (cf. [22] (4.4)):

$$\int_0^{\pi/2} \log \cos \theta d\theta = -\frac{\pi}{2} \log 2,$$

we have

$$\begin{aligned} \int_0^\pi \log \sin \theta \frac{d\theta}{\pi} &= 2 \int_0^{\pi/2} \log \sin \theta \frac{d\theta}{\pi} \\ &= 2 \int_0^{\pi/2} \log \cos\left(\frac{\pi}{2} - \theta\right) \frac{d\theta}{\pi} \\ &= 2 \int_0^{\pi/2} \log \cos \theta \frac{d\theta}{\pi} = -\log 2. \end{aligned}$$

Thus by (5.7), we have $V_{\mathbb{T}}(1) = 0$. Since g_1 is radial, we see that

$$V_{\mathbb{T}}(e^{i\theta}) = 0, \quad \forall \theta \in [0, 2\pi].$$

6. POWER SET AT ISOLATED SPECTRAL POINT

Suppose V is a densely defined linear operator on \mathcal{H} whose spectrum $\sigma(V)$ contains an isolated spectral point. Without loss of generality we may assume 0 is an isolated spectral point, and $\delta > 0$ is such that the punctured disk $D_\delta = \{0 < |z| < \delta\}$ contains no spectral point, e.g. $D_\delta \subset \rho(V)$. Then for every unit vector $x \in \mathcal{H}$ the metric

$$g_x(z) = \|(V - z)^{-1}x\|^2, \quad 0 < |z| < \delta,$$

is well-defined on D_δ . Depending on x , the metric function $g_x(z)$ may or may not blow up at $z = 0$. In the case $g_x(z)$ does blow up at $z = 0$, the blow up rate varies with respect to the choice of x . In this and the next section we will study the blow up rate of $g_x(z)$ at $z = 0$, define *power set* and reveal its connection with the hyper-invariant subspaces of V .

One way to measure the blow up rate of $g_x(z)$ is to compare it with that of the function $g(z) = \|(V - z)^{-1}\|^2$. Note that $g(z)$ also defines a metric on D_δ . This comparison can be made through the length of the circles $C_r = \{z : 0 < |z| = r < \delta\}$

$\delta\}$ with respect to the two metrics. For the parametrization $z(t) = re^{2\pi it}$, $0 \leq t \leq 1$, the length of the circle C_r with respect to metric g_x is given by

$$L_x(C_r) = \int_0^1 |z'(t)| \|(V - z(t))^{-1}x\| dt = 2\pi r \int_0^1 \|(V - re^{2\pi it})^{-1}x\| dt, \quad (6.1)$$

and that with respect to g is

$$L(C_r) = \int_0^1 |z'(t)| \|(V - z(t))^{-1}\| dt = 2\pi r \int_0^1 \|(V - re^{2\pi it})^{-1}\| dt. \quad (6.2)$$

Now we define

$$P_0 = \frac{1}{2\pi i} \int_{C_r} (V - z)^{-1} dz, \quad \text{and} \quad V_0 = \frac{1}{2\pi i} \int_{C_\eta} z(V - z)^{-1} dz. \quad (6.3)$$

Clearly, if V is quasi-nilpotent, e.g. V is bounded and $\sigma(V) = \{0\}$, then $P_0 = I$ and $V = V_0$ by functional calculus. But in general P_0 and V_0 are only a “part” of I and respectively V . It follows from (5.5) that $L_x(C_r) \geq 2\pi\|P_0x\|$ no matter how small r is. We make the following definition to proceed.

Definition. Assume 0 is an isolated spectral point of a densely defined linear operator V on \mathcal{H} , and $x \in \mathcal{H}$. 0 is said to be an essential singularity with respect to the metric g_x if there is no natural number N such that $r^N L_x(C_r)$ is bounded for all positive r in a neighborhood of 0.

Lemma 6.1. *If 0 is a non-essential isolated spectral point of V with respect to g_x , then there is a natural number N such that $V_0^N x = 0$.*

Proof. Since 0 is non-essential, there exists $N \in \mathbb{N}$, $M > 0$ and $0 < \delta_x < \delta$ such that $r^{N-1} L_x(C_r) \leq M$ for all $0 < r \leq \delta_x$. Then for every λ with $0 < \eta := 2|\lambda| < \delta_x$, we use functional calculus to write

$$(V - \lambda)^{-1} = \frac{1}{2\pi i} \int_{C_\eta} \frac{(V - z)^{-1}}{z - \lambda} dz. \quad (6.4)$$

Then under the parametrization $z(t) = \eta e^{2\pi i t}$, $0 \leq t \leq 1$, we use (6.4) to write

$$\begin{aligned}
|\lambda|^N \|(V - \lambda)^{-1}x\| &= |\lambda|^N \left\| \frac{1}{2\pi i} \int_0^1 \frac{(V - z(t))^{-1}x}{z(t) - \lambda} z'(t) dt \right\| \\
&\leq \frac{|\lambda|^N}{2\pi} \int_0^1 \frac{\|(V - z(t))^{-1}x\|}{|z(t) - \lambda|} |z'(t)| dt \\
&\leq \frac{|\lambda|^N}{2\pi(\eta - |\lambda|)} \int_0^1 \|(V - z(t))^{-1}x\| |z'(t)| dt \\
&= \frac{\eta^{N-1}}{2^N \pi} L_x(C_\eta) \\
&\leq \frac{M}{2^N \pi}.
\end{aligned}$$

This shows that $|z|^N \|(V - z)^{-1}x\|$ is bounded on the punctured disk $D_{\delta_x/2} = \{0 < |z| < \frac{\delta_x}{2}\}$. For every bounded linear functional $y^* \in \mathcal{H}^*$, we consider the Laurent series

$$y^*((V - z)^{-1}x) = \sum_{j=-\infty}^{\infty} c_j z^j, \quad z \in D_{\delta_x/2}.$$

The boundedness of $|z|^N \|(V - z)^{-1}x\|$ clearly implies the boundedness of $z^N y^*((V - z)^{-1}x)$ on $D_{\delta_x/2}$ for every fixed y^* . Hence $c_j = 0$ for $j \leq -N - 1$ by complex analysis. By functional calculus we have

$$\begin{aligned}
0 = c_{-N-1} &= \frac{1}{2\pi i} \int_{C_\eta} z^N y^*((V - z)^{-1}x) dz \\
&= y^* \left(\frac{1}{2\pi i} \int_{C_\eta} z^N (V - z)^{-1} x dz \right) \\
&= y^*(V_0^N x).
\end{aligned}$$

Since y^* is arbitrary, it follows that $V_0^N x = 0$. □

In view of Lemma 6.1, the principal part of $(V - z)^{-1}x$ at 0 is

$$\frac{V_0^{N-1}x}{z^N} + \frac{V_0^{N-2}x}{z^{N-1}} + \cdots + \frac{V_0 x}{z^2} + \frac{P_0 x}{z}. \quad (6.5)$$

It is easy to see that $L_x(C_r) \leq L(C_r)$ for every r and every x such that $\|x\| = 1$. So if $r^{N-1}L(C_r)$ is bounded for all r close to 0, then by Lemma 6.1 we have $V_0^N x = 0$ for every x , and hence the following

Corollary 6.2. *If there is $N \in \mathbb{N}$ such that $r^{N-1}L(C_r)$ is bounded for all r close to 0, then $V_0^N = 0$.*

As indicated in Lemma 6.1, the blow-up rate of $L_x(C_r)$ as $r \rightarrow 0$ is a gauge of the action of V on x . This blow-up occurs because g_x has a singularity at 0. For a more in-depth study, we define

$$k_x = \limsup_{|z| \rightarrow 0} \frac{\log g_x(z)}{\log g(z)}.$$

Since $g_x(z) \leq g(z)$, we clearly have $k_x \leq 1$. Moreover, since $g(z)$ goes to ∞ as $|z| \rightarrow 0$, for any non-zero constant c we have $g_{cx}(z) = |c|^2 g_x(z)$ which implies $k_x = k_{cx}$. Further, by

$$\|V - z\| \sqrt{g_x(z)} = \|V - z\| \|(V - z)^{-1}x\| \geq \|x\| = 1,$$

we have $g_x(z) \geq \|V - z\|^{-2}$ and hence

$$\limsup_{|z| \rightarrow 0} \frac{\log g_x(z)}{\log g(z)} \geq \limsup_{|z| \rightarrow 0} \frac{-2 \log \|V - z\|}{\log g(z)} = 0.$$

This concludes that $0 \leq k_x \leq 1$. Since, roughly speaking, as $z \rightarrow 0$ the metric function $g_x(z)$ blows up at most as fast as $g(z)^{k_x}$, the relative blow up rate of $g_x(z)$ is gauged by the power k_x .

Definition. Let V be a bounded linear operator on a Banach space \mathcal{H} with isolated spectral point 0. The set $\Lambda(V) := \{k_x : x \in \mathcal{H}, x \neq 0\}$ shall be called power set of V at 0.

The power set has spectral properties in the following sense.

Proposition 6.3. *If V_1 and V_2 are similar and have isolated spectral point at 0, then $\Lambda(V_1) = \Lambda(V_2)$.*

Proof. Let δ be small such that D_δ contains no other spectral points, and assume $|z| < \delta$. Suppose S is an invertible operator such that $S^{-1}V_1S = V_2$. Then we have

$$\|(V_2 - z)^{-1}x\| = \|S^{-1}(V_1 - z)^{-1}Sx\| \leq \|S^{-1}\| \|(V_1 - z)^{-1}Sx\|,$$

and

$$\|(V_2 - z)^{-1}\| = \|S^{-1}(V_1 - z)^{-1}S\| \geq \frac{\|(V_1 - z)^{-1}\|}{\|S\|\|S^{-1}\|}. \quad (6.6)$$

If $\|(V_1 - z)^{-1}Sx\|$ is bounded near $z = 0$, then since both $\|(V_1 - z)^{-1}\|$ and $\|(V_2 - z)^{-1}\|$ tend to ∞ as $|z| \rightarrow 0$, we easily have $k_x(V_2) = k_{Sx}(V_1) = 0$.

Now we assume $\|(V_1 - z)^{-1}Sx\|$ is unbounded near $z = 0$. Then for all nonzero z close enough to 0, using (6.6) we have

$$\frac{\log \|(V_2 - z)^{-1}x\|}{\log \|(V_2 - z)^{-1}\|} \leq \frac{\log \|S^{-1}\| + \log \|(V_1 - z)^{-1}Sx\|}{\log \|(V_1 - z)^{-1}\| - \log(\|S\|\|S^{-1}\|)}.$$

Taking \limsup as $|z| \rightarrow 0$ on both sides, we have that $k_x(V_2) \leq k_{Sx}(V_1)$.

For the other direction, by (6.6) we have

$$\|(V_2 - z)^{-1}x\| = \|S^{-1}(V_1 - z)^{-1}Sx\| \geq \frac{\|(V_1 - z)^{-1}Sx\|}{\|S\|}, \quad (6.7)$$

and

$$\|(V_2 - z)^{-1}\| \leq \|S^{-1}\|\|S\|\|(V_1 - z)^{-1}\|.$$

It follows that

$$\frac{\log \|(V_2 - z)^{-1}x\|}{\log \|(V_2 - z)^{-1}\|} \geq \frac{\log \|(V_1 - z)^{-1}Sx\| - \log \|S\|}{\log \|(V_1 - z)^{-1}\| + \log(\|S\|\|S^{-1}\|)},$$

and taking \limsup as $|z| \rightarrow 0$ we have $k_x(V_2) \geq k_{Sx}(V_1)$. In conclusion, $k_x(V_2) = k_{Sx}(V_1)$ for every $x \neq 0$, and this in particular shows $\Lambda(V_1) = \Lambda(V_2)$. \square

Proposition 6.3 shows that power set is invariant under similarity of operators.

In the rest of this section we shall compute the power set for two most well-known examples of quasi-nilpotent operators. A bounded linear operator V is said

to be nilpotent if there is a positive integer n such that $V^n = 0$, and V is quasi-nilpotent if $\sigma(V) = \{0\}$. Compact quasi-nilpotent operators has been well studied in [20], where a classical example is the Volterra operator. In the rest of this section we shall compute the power set for nilpotent operators and for the classical Volterra operators. In the next section we shall point out a natural connection between power set and hyper-invariant subspaces of quasinilpotent operators.

Observe that in this case the metric function $g_x(z) = \|(V - zI)^{-1}x\|^2$ is defined on the punctured complex plane $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Example 6.4. *If V is nilpotent with $V^n = 0$ but $V^{n-1} \neq 0$, where $n \geq 1$, then setting $w = \frac{1}{z}$, we have*

$$(V - z)^{-1} = -w(I - wV)^{-1} = -w(I + wV + \cdots + w^{n-1}V^{n-1}). \quad (6.8)$$

Let y be such that $V^{n-1}y \neq 0$. For $1 \leq k \leq n$ if we set $x = V^{n-k}y$, then $V^{k-1}x \neq 0$ but $V^kx = 0$. Hence by (6.8)

$$(V - z)^{-1}x = -(wx + w^2Vx + \cdots + w^kV^{k-1}x).$$

Hence as $|z| \rightarrow 0$, the principle part of $g_x(z)$ is $\frac{\|V^{k-1}x\|^2}{|z|^{2k}}$, and by (6.8) the principle part of $g(z)$ is $\frac{\|V^{n-1}\|^2}{|z|^{2n}}$. Therefore

$$k_x = \limsup_{|z| \rightarrow 0} \frac{\log g_x(z)}{\log g(z)} = \limsup_{|z| \rightarrow 0} \frac{\log(\|V^{k-1}x\|/|z|^k)}{\log(\|V^{n-1}\|/|z|^n)} = \frac{k}{n}.$$

This shows that $\Lambda(V) = \{\frac{k}{n} : 1 \leq k \leq n\}$.

Now we take a look at the classical Volterra operator $V : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined by

$$Vf(x) = \int_0^x f(t)dt.$$

It is known ([15]) that

$$V^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt, \quad n \geq 1$$

so setting $w = \frac{1}{z}$ we have

$$\begin{aligned} (V - z)^{-1} f(x) &= -w(I - wV)^{-1} f(x) \\ &= -w \sum_{n=0}^{\infty} w^n V^n f(x) \\ &= -w \left(f(x) + \int_0^x \sum_{n=1}^{\infty} \frac{w^n (x-t)^{n-1}}{(n-1)!} f(t) dt \right) \\ &= -w \left(f(x) + w \int_0^x e^{w(x-t)} f(t) dt \right). \end{aligned} \quad (6.9)$$

For $0 \leq \alpha < 1$, we let f_α be the characteristic function for the interval $[\alpha, 1]$. Then is not hard to check by (6.9) that $(V - z)^{-1} f_\alpha(x) = 0$, for $0 \leq x \leq \alpha$, and for $x > \alpha$

$$(V - z)^{-1} f_\alpha(x) = -w - w^2 \int_\alpha^x e^{w(x-t)} dt = -w e^{w(x-\alpha)}.$$

Therefore,

$$\begin{aligned} \|(V - z)^{-1} f_\alpha\|^2 &= |w|^2 \int_\alpha^1 |e^{w(x-\alpha)}|^2 dx \\ &= |w|^2 \int_\alpha^1 e^{(w+\bar{w})(x-\alpha)} dx = (e^{\frac{(z+\bar{z})}{|z|^2}(1-\alpha)} - 1) / (z + \bar{z}). \end{aligned}$$

Observe that when $|z|$ tends to 0,

$$\begin{aligned} 2 \log \|(V - z)^{-1} f_\alpha\| &= \log |e^{\frac{(z+\bar{z})}{|z|^2}(1-\alpha)} - 1| - \log |z + \bar{z}| \\ &= O\left(\frac{z + \bar{z}}{|z|^2}(1-\alpha)\right) - \log |z + \bar{z}| \\ &= O\left(\frac{z + \bar{z}}{|z|^2}(1-\alpha)\right) = O\left(\frac{2 \cos \theta}{|z|}(1-\alpha)\right), \end{aligned} \quad (6.10)$$

where $z = |z|e^{i\theta}$. On the other hand, by [26],

$$\frac{1}{2n!} \leq \|V^n\| \leq \frac{1}{2n!} \left(1 - \frac{1}{2n}\right)^{-1/2}, \quad n \geq 1.$$

Since $(1 - \frac{1}{2n})^{-1/2} \leq \sqrt{2}$, we have $\|V^n\| \leq \frac{1}{\sqrt{2n!}}$ for $n \geq 1$. Combining with the expansion

$$(V - z)^{-1} = -w(I - wV)^{-1} = -w \sum_{n=0}^{\infty} w^n V^n,$$

where $w = 1/z$, we therefore have

$$\begin{aligned} \|(V - z)^{-1}\| &\leq |w| \sum_{n=0}^{\infty} |w|^n \|V^n\| \\ &\leq |w| \left(1 + \sum_{n=1}^{\infty} \frac{|w|^n}{\sqrt{2n!}} \right) \\ &= |w| \left(\frac{1}{\sqrt{2}} e^{|w|} + \left(1 - \frac{1}{\sqrt{2}}\right) \right) \\ &\leq |w| \left(\frac{1}{\sqrt{2}} e^{|w|} + \left(1 - \frac{1}{\sqrt{2}}\right) e^{|w|} \right) \\ &\leq |w| e^{|w|} = \frac{1}{|z|} e^{1/|z|}. \end{aligned} \tag{6.11}$$

Note that $f_0 = 1$ and $\|(V - z)^{-1}\| \geq \|(V - z)^{-1} f_0\|$, we therefore have the inequalities

$$\begin{aligned} \frac{\log \|(V - z)^{-1} f_{\alpha}\|}{\log(\frac{1}{|z|} e^{1/|z|})} &\leq \frac{\log \|(V - z)^{-1} f_{\alpha}\|}{\log \|(V - z)^{-1}\|} \\ &\leq \frac{\log \|(V - z)^{-1} f_{\alpha}\|}{\log \|(V - z)^{-1} f_0\|}. \end{aligned} \tag{6.12}$$

For the first term above, $\log(\frac{1}{|z|} e^{1/|z|}) = O(\frac{1}{|z|})$, and we use (6.10) to write

$$\limsup_{|z| \rightarrow 0} \frac{\log \|(V - z)^{-1} f_{\alpha}\|}{\log(\frac{1}{|z|} e^{1/|z|})} = \limsup_{|z| \rightarrow 0} \frac{\frac{\cos \theta}{|z|} (1 - \alpha)}{\frac{1}{|z|}} = 1 - \alpha.$$

For the last term in (6.12), using (6.10) we have

$$\limsup_{|z| \rightarrow 0} \frac{\log \|(V - z)^{-1} f_{\alpha}\|}{\log \|(V - z)^{-1} f_0\|} = \limsup_{|z| \rightarrow 0} \frac{\frac{2 \cos \theta}{|z|} (1 - \alpha)}{\frac{2 \cos \theta}{|z|}} = 1 - \alpha.$$

We therefore conclude that for the classical Volterra operator V we have

$$k_{f_{\alpha}} = 1 - \alpha, \quad 0 \leq \alpha < 1. \tag{6.13}$$

In summary, $(0, 1] \subset \Lambda(V)$. Later we shall see that the inclusion is in fact an equality.

7. HYPER-INVARIANT SUBSPACES

The invariant subspace problem for quasi-nilpotent operators has been a tempting but very difficult one. We refer the readers to [1, 11, 17, 19] and the references therein for more information. This section makes no particular attempt on this problem, but it shall point out a natural connection between hyperinvariant subspaces and the power set. This connection in fact holds for every bounded linear operator V on a Banach space \mathcal{H} with an isolated spectral point at 0.

Let g, g_x, k_x and $\Lambda(V)$ be defined as in Section 6. For any $0 \leq \tau \leq 1$, we set

$$M_\tau = \{x \in \mathcal{H} : k_x \leq \tau\} \cup \{0\}.$$

Theorem 7.1. *M_τ is a hyper-invariant subspace for V .*

Proof. We first check M_τ is a linear space. It is easy to see that if $x \in M_\tau$, then $cx \in M_\tau$ for any complex number c . If x and y are in M_τ . Then $\forall \epsilon > 0$, there exist $\delta > 0$ such that $D_\delta \subset \rho(V)$ and for all z with $|z| < \delta$, we have

$$\frac{\log g_x(z)}{\log g(z)} < \tau + \epsilon, \text{ and } \frac{\log g_y(z)}{\log g(z)} < \tau + \epsilon, \quad (7.1)$$

or equivalently,

$$\|(V - z)^{-1}x\| < \|(V - z)^{-1}\|^{\tau+\epsilon}, \text{ and } \|(V - z)^{-1}y\| < \|(V - z)^{-1}\|^{\tau+\epsilon}.$$

It follows that

$$\begin{aligned} \log \|(V - z)^{-1}(x + y)\| &\leq \log (\|(V - z)^{-1}x\| + \|(V - z)^{-1}y\|) \\ &\leq \log (2\|(V - z)^{-1}\|^{\tau+\epsilon}) \\ &= \log 2 + (\tau + \epsilon) \log \|(V - z)^{-1}\|, \end{aligned}$$

and hence

$$\frac{\log \|(V - z)^{-1}(x + y)\|}{\log \|(V - z)^{-1}\|} \leq \tau + \epsilon + \frac{2}{\log \|(V - z)^{-1}\|}.$$

Since $\log \|(V - z)^{-1}\|$ goes to ∞ as z goes to 0, taking \limsup the above inequality implies $k_{x+y} \leq \tau + \epsilon$. Since $\epsilon > 0$ is arbitrary, this proves $k_{x+y} \leq \tau$ e.g. $x + y \in M_\tau$.

Next we check that M_τ is closed. Assume $\{x_n\}$ is a sequence in M_τ such that $x_n \rightarrow y$ in norm and $\|x_n\| = 1$ for each n . First pick a $\delta' < \min\{\|V\|, \delta\}$ such that $\|(V - z)^{-1}\| \geq e^2$, $\forall |z| < \delta'$. For any fixed $\epsilon > 0$, there exists z' such that $|z'| < \delta'$ and

$$k_y - \epsilon < \frac{\log g_y(z')}{\log g(z')}. \quad (7.2)$$

Since $x_n \rightarrow y$ in norm, we can pick a particular x_n such that $\|(V - z')^{-1}(x_n - y)\| < \epsilon$. Then using inequality $\log(1 + t) \leq t$, $t > -1$, one computes that

$$\begin{aligned} \frac{\log g_y(z')}{\log g(z')} &= \frac{\log \|(V - z')^{-1}(x_n + y - x_n)\|}{\log \|(V - z')^{-1}\|} \\ &\leq \frac{\log (\|(V - z')^{-1}x_n\| + \|(V - z')^{-1}(x_n - y)\|)}{\log \|(V - z')^{-1}\|} \\ &= \frac{\log \|(V - z')^{-1}x_n\|}{\log \|(V - z')^{-1}\|} + \frac{\log (1 + \|(V - z')^{-1}x_n\|^{-1}\|(V - z')^{-1}(x_n - y)\|)}{\log \|(V - z')^{-1}\|} \\ &\leq \frac{\log \|(V - z')^{-1}x_n\|}{\log \|(V - z')^{-1}\|} + \frac{\|(V - z')^{-1}x_n\|^{-1}\|(V - z')^{-1}(x_n - y)\|}{\log \|(V - z')^{-1}\|}. \end{aligned} \quad (7.3)$$

Since

$$\|(V - z')^{-1}x_n\| \geq \frac{1}{\|V - z'\|},$$

we have

$$\|(V - z')^{-1}x_n\|^{-1} \leq \|V - z'\| \leq 2\|V\|.$$

Further, since $|z'| < \delta'$, we have $\|(V - z')^{-1}\| > e^2$. By (7.2) and (7.3) we then have

$$k_y - \epsilon < \frac{\log g_y(z')}{\log g(z')} \leq \frac{\log \|(V - z')^{-1}x_n\|}{\log \|(V - z')^{-1}\|} + \|V\|\epsilon.$$

It implies

$$k_y - \epsilon < \sup_{|z| < \delta'} \frac{\log g_{x_n}(z)}{\log g(z)} + \|V\|\epsilon.$$

Taking limit as $\delta' \rightarrow 0$, we have $k_y - \epsilon \leq k_{x_n} + \|V\|\epsilon$, and it implies $k_y \leq k_{x_n} \leq \tau$ since ϵ is arbitrarily. Therefore $y \in M_\tau$, and this concludes that M_τ is a closed subspace of \mathcal{H} .

Finally, if A is a bounded linear operator that commutes with V , and $x \in M_\tau$, then

$$\begin{aligned} k_{Ax} &= \limsup_{|z| \rightarrow 0} \frac{\log \|(V - z)^{-1}Ax\|}{\log g(z)} \\ &= \limsup_{|z| \rightarrow 0} \frac{\log \|A(V - z)^{-1}x\|}{\log g(z)} \\ &\leq \limsup_{|z| \rightarrow 0} \frac{\log \|A\| + \log \|(V - z)^{-1}x\|}{\log g(z)} \\ &= k_x \leq \tau. \end{aligned}$$

Hence $Ax \in M_\tau$. This concludes that M_τ is a hyper-invariant subspace of V . \square

Some corollaries are immediate.

Corollary 7.2. *If $\Lambda(V)$ contains at least two points then V has a nontrivial hyper-invariant subspace.*

Proof. Suppose there are non-zero x and y such that $k_x < k_y$, then set $M = M_{k_x}$. Since $x \in M$ and $y \notin M$, M is a nontrivial hyper-invariant subspace. \square

So in particular if V is quasi-nilpotent and $\Lambda(V)$ contains at least two points then V has a nontrivial hyper-invariant subspace! An immediate consequence of the proof of Corollary 7.2 is that if x and y are both cyclic for V , then $k_x = k_y$, and this value is maximal in $\Lambda(V)$. We state this observation as

Corollary 7.3. *If V has a cyclic vector x , then $k_x = \max \Lambda(V)$.*

The converse of Corollary 7.3 is not true. For example, consider the direct sum $V' = V \oplus V$ on $L^2([0, 1]) \otimes \mathbb{C}^2$, where V is the Volterra operator. If $x = (1, 0)$, then $k_x(V') = k_1(V) = 1$ by (6.13), which is maximal, but $(1, 0)$ is clearly not cyclic for V' . However, the following two questions seem natural.

Question 2. If V is irreducible (e.g. having no nontrivial reducing subspace) and $k_x = \sup \Lambda(V)$, then is x cyclic for V ?

Question 3. Is $\sup \Lambda(V) = 1$ for all quasi-nilpotent V ?

It is not hard to verify that if $V = 0$, then $\Lambda(V) = \{1\}$. The last question is related to the invariant subspace problem for quasi-nilpotent operators.

Question 4. Is there a nontrivial quasi-nilpotent operator V such that $\Lambda(V)$ is a single point?

Clearly, if the answer is no, then Corollary 7.2 proves every quasi-nilpotent operator has a nontrivial hyper-invariant subspace. If the answer is yes, then characterizing those quasi-nilpotent operators V with a singleton $\Lambda(V)$ becomes an appealing problem.

In Section 6, we have computed that $\Lambda(V) = (0, 1]$ for the classical Volterra operator. If there were non-zero f such that $k_f = 0$, then by Theorem 7.1, M_0 is a nontrivial hyper-invariant subspace for V . Then, as a well-known result, M_0 is of the form $f_\alpha L^2$ for some $\alpha > 0$. But by (6.18), we have $k_{f_\alpha} = 1 - \alpha > 0$, which is a contradiction. Now we summarize our computations about the classical Volterra operator.

Corollary 7.4. *For classical Volterra operator V , the power set $\Lambda(V) = (0, 1]$.*

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